Necessary conditions for a weak minimum in optimal control problems with integral equations on a variable time interval

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We study an optimal control problem with Volterra-type integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f(t, s, x(s), u(s))\, ds,$$

considered on a nonfixed time interval $[t_0, t_1]$, subject to endpoint constraints of equality and inequality type, and the cost in the Mayer form. We obtain first-order necessary optimality conditions for an extended weak minimum, the notion of which is a natural generalization of the notion of weak minimum with account of variations of the time. The conditions obtained generalize the Euler–Lagrange equation and transversality conditions for the Lagrange problem in the classical calculus of variations with ODEs. Their novelty, as compared with those for problems on a fixed time interval is that the costate equation and transversality condition with respect to time variable involve nonstandard terms that are absent in problems with ODEs.

To be more precise, introducing the the costate variables ψ_x and ψ_t corresponding to x and t, respectively, we obtain the costate equations

$$\dot\psi_x(s) = \psi_x(s)\, f_x(s, s, x^0(s), u^0(s)) + \int_s^{\hat t_1} \psi_x(t)\, f_{tx}(t, s, x^0(s), u^0(s))\, dt,$$

$$\dot\psi_t(s) = \psi_x(s)\, f_s(s, s, x^0(s), u^0(s)) + \int_s^{\hat t_1} \psi_x(t)\, f_{ts}(t, s, x^0(s), u^0(s))\, dt - \dot\psi_x(s)\, F(s),$$

where $F(s) = \int_{t_0}^{s} f_t(s, z, x^0(z), u^0(z))\, dz$, and f_t, f_s denote the derivatives w.r.t. first and second variables of f, respectively, the transversality conditions

$$\psi_x(\hat t_0) = l_{x_0}, \qquad \psi_x(\hat t_1) = -l_{x_1},$$
$$\psi_t(\hat t_0) = l_{t_0}, \qquad \psi_t(\hat t_1) = -l_{t_1} + \psi_x(\hat t_1)\, F(\hat t_1),$$

where $l(t_0, x_0, t_1, x_1)$ is the usual endpoint Lagrange function, the "energy evolution law"

$$\psi_t(s) + \psi_x(s)\, f(s, s, x^0(s), u^0(s)) + \int_s^{\hat t_1} \psi_x(t)\, f_t(t, s, x^0(s), u^0(s))\, dt = 0,$$

which, in fact, determines ψ_t via ψ_x, and the stationarity condition with respect to the control

$$\psi_x(s)\, f_u(s, s, x^0(s), u^0(s)) + \int_s^{\hat t_1} \psi_x(t)\, f_{tu}(t, s, x^0(s), u^0(s))\, dt = 0.$$

The proof is based on reducing the problem to a problem on a fixed time interval and then using the general Lagrange multipliers rule. However, in contrast to problems with ODEs, this reducing leads to an integral equation which is not of the standard Volterra-type, and this is the cause of appearance of the new additional terms.