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Recent results on analysis and optimal control of phase field models

Cahn–Hilliard approach to Stefan problem with dynamic boundary condition

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Abstract: In this talk, recent advances in the equations and dynamic boundary conditions of a sort of Cahn–Hilliard type is introduced. Moreover, based on these results, the existence problem for the two phase Stefan problem with dynamic boundary condition is discussed.

Let $0 < T < +\infty$ and $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be a bounded smooth domain occupied by a material. Also the boundary Γ of Ω is supposed to be smooth enough. Start from the following equations of Cahn–Hilliard type in the domain $Q := (0,T) \times \Omega$: For all $\varepsilon > 0$

$$\frac{\partial u}{\partial t} - \Delta \mu = 0 \quad \text{in } Q, \tag{1}$$

$$\mu = -\varepsilon \Delta u + \beta(u) + \varepsilon \pi(u) - g \quad \text{in } Q, \tag{2}$$

where unknowns $u, \mu : Q \to \mathbb{R}$ stand for the order parameter, the chemical potential, respectively. In order to consider the dynamics on the boundary $\Sigma := (0,T) \times \Gamma$, also find unknowns $u_\Gamma, \mu_\Gamma : \Sigma \to \mathbb{R}$ satisfying $u_\Gamma = u_{|\Gamma}$, $\mu_\Gamma = \mu_{|\Gamma}$ on Σ, where $u_{|\Gamma}$ and $\mu_{|\Gamma}$ denote the trace of u and μ, respectively, and consider the same type of equations on the boundary

$$\frac{\partial u_\Gamma}{\partial t} + \partial_\nu \mu - \Delta_\Gamma \mu_\Gamma = 0 \quad \text{on } \Sigma, \tag{3}$$

$$\mu_\Gamma = \varepsilon \partial_\nu u - \varepsilon \Delta_\Gamma u_\Gamma + \beta_\Gamma(u_\Gamma) + \varepsilon \pi_\Gamma(u_\Gamma) - g_\Gamma \quad \text{on } \Sigma, \tag{4}$$

where ∂_ν represents the outward normal derivative on Γ; Δ_Γ stands for the Laplace–Beltrami operator on Γ. This kind of dynamic boundary condition (3)–(4) is a sort of transmission problem between the dynamics in the bulk Ω and one on the boundary Γ. Together with the initial conditions $u(0) = u_0$ in Ω, $u_\Gamma(0) = u_{0\Gamma}$ on Γ, the system (1)–(4) was introduced by Goldstein, Miranville and Schimperna (2011), and studied by many authors from the various view points. The prototype as the Cahn–Hilliard model is provided by $\beta(r) = \beta_\Gamma(r) = r^3$ and $\pi(r) = \pi_\Gamma(r) = -r$ for all $r \in \mathbb{R}$ and the choice of these functions is extended to the singular potentials. In order to treat the degenerate parabolic equation, more precisely two phase Stefan problem, we consider the passing to the limit $\varepsilon \to 0$.