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Some aspects of Variational Analysis and Applications

Gradient projection method for convex functions and strongly convex sets

M. O. Golubev

Moscow Institute of Physics and Technology (State University)

maksimkane@mail.ru

Abstract: Let $\mathbb{H}$ be the real Hilbert space. $\langle p, x\rangle$ is scalar product for vectors $p, x \in \mathbb{H}$. Let $B_R(x) = \{y \in \mathbb{H} : \|y - x\| \leq R\}$. We denote the boundary of the set $A \subset \mathbb{H}$ by ∂A. The metric projection of the point $x \in \mathbb{H}$ on the set $A \subset \mathbb{H}$ is defined as follows: $P_A(x) = \{y \in A | \|x - y\| = \inf_{a\in A} \|x - a\|\}$. We denote the normal cone to the closed convex set A at the point $a \in A$ by $N(A; a)$, i.e. $N(A; a) = \{p \in \mathbb{H} : \langle p, a\rangle = \sup_{x\in A}\langle p, x\rangle\}$.

Definition. [1, Definition 4.3.1] A nonempty set $A \subset \mathbb{H}$ is called *strongly convex of radius* R, if it can be represented as the intersection of closed balls of radius $R > 0$.

Consider the minimization problem: $\min_{x\in A} f(x)$. We consider the standard gradient projection algorithm: $x_1 \in \partial A$, $x_{k+1} = P_A(x_k - \alpha_k f'(x_k))$, $\quad \alpha_k > 0$. (1)

Suppose that:

(i) $A \subset \mathbb{H}$ is strongly convex with radius R,

(ii) $f : \mathbb{H} \to \mathbb{R}$ is convex, differentiable and the gradient $f'(x)$ satisfies the Lipschitz condition with constant $M > 0$,

(iii) for any $k \in \mathbb{N}$ there exists a unit vector $n(x_k) \in N(A; x_k)$ such that $\langle n(x_k), f'(x_k)\rangle \leq 0$,

(iv) the problem (1) has a unique solution $x_* \in \partial A$.

Theorem 1. Suppose that conditions (i)-(iv) hold. Then for $\alpha_k = \alpha \in \left(0, \frac{2}{M}\right]$ we have the following estimate of the convergence rate of the algorithm:

$$\|x_{k+1} - x_*\| \leq \sqrt[4]{\frac{R^2}{R^2 + \alpha^2\|f'(x_k)\|^2}}\|x_k - x_*\|.$$

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1. *E. S. Polovinkin, M. V. Balashov.* Elements of convex and strongly convex analysis. Fizmatlit, Moscow, 2007 (in Russian).