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Some aspects of Variational Analysis and Applications

Separation theorems for weakly convex sets in Banach spaces with nonsymmetric seminorm[1]

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Abstract: A quasiball in a Banach space E is a convex closed set $M \subset E$ such that $0 \in \operatorname{int} M$ and $M \neq E$. The Minkowski functional $\mu_M(x) = \inf \left\{ t > 0 \middle| \; x \in tM \right\}$ of the quasiball is the nonsymmetric seminorm. The *M-distance* from a set C to a set A is $\varrho_M(C, A) = \inf\limits_{c\in C,\ a\in A} \mu_M(c-a)$. The *$M$-projection* of x onto A is the set $P_M(x, A) = A \bigcap (x - \varrho_M(x, A)M)$. The set $C \subset E$ is called *strongly convex with respect to the quasiball* $M \subset E$ if C is convex, closed and there exists a set $C_1 \subset E$ such that $C + C_1 = M$. A set $A \subset E$ is called *weakly convex* with respect to the quasiball $M \subset E$ if $a \in P_M(a + z, A), \quad \forall a \in A, \quad \forall z \in N^1_M(a, A)$, where $N^1_M(a, A) = \{z \in \partial M \mid \exists t > 0 : \; a \in P_M(a + tz, A)\}$. A set $M \subset E$ is called *parabolic*, if for any vector $b \in E$ the set $\left(b + \frac{1}{2}M\right) \setminus M$ is bounded. A set $M \subset E$ is called *boundedly uniformly convex*, if it is convex and $\lim_{t\to+0} \delta_M(t, R) = 0$ for any $R > 0$, where $\delta_M(t, R) = \sup \left\{ \|a - b\| \;\middle|\; a, b \in M \cap \mathfrak{B}_R(0),\ \inf_{x\in\partial M} \left\| \frac{a+b}{2} - x \right\| < t \right\}, \quad t \geq 0.$

Theorem 1 *Let E be a Banach space and the quasiball $M \subset E$ be parabolic and boundedly uniformly convex. Let $0 < r < R$, the sets $A, C \subset E$ be closed, A be weakly convex with respect to the set RM, C be strongly convex with respect to the set $(-rM)$, $A + R \operatorname{int} M \neq E$. Let at least one of the following statements hold*
1) $\varrho_M(C, A) > 0$ or
2) $\operatorname{int} C \neq \emptyset$, $A \cap \operatorname{int} C = \emptyset$ and the quasiball M is uniformly smooth, the set A is M-quasibounded, i.e. for any point $x \in E \setminus A$ we have $\varrho_M(x, A) > 0$ and for any $R > 0$ the inequality $\sup_{a\in\partial A\cap\mathfrak{B}_R(0)} \sup\limits_{z\in N^1_M(a,A)} \|z\| < +\infty$ holds.
Then there exist $a, c \in E$ such that $\operatorname{int} C \subset c - \operatorname{int} rM \subset a - \operatorname{int} RM \subset E \setminus A$.

This theorem is an analog of the famous Hahn-Banach separation theorem. Such an approach allows us to apply the methods of proximal analysis to the epigraphs of functions and to obtain the conditions of well-posedness for optimization problems of the infimal convolution type. The result was obtained under the supervision of professor G.E. Ivanov.

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