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Some aspects of Variational Analysis and Applications

Hypomonotonicity of the normal cone and proximal smoothness[1]

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Abstract: Let X be a real Banach space. We use $\langle p, x\rangle$ to denote the value of functional $p \in X^*$ at the vector $x \in X$. Let $\rho_X(\tau) = \sup\left\{\frac{\|x+y\|}{2} + \frac{\|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}$. The function $\rho_X(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is referred to as the module of smoothness of X. We denote by $\mathfrak{B}_R(c)$ a closed ball with center c and radius R.
A set $A \subset X$ is said to be proximally smooth with constant R if the distance function $x \to \rho(x, A)$ is continuously differentiable on set $U(R, A) = \{x \in X : 0 < \rho(x, A) < R\}$.
By $\mathfrak{M}$ denote the set of convex and Lipschitz continuous functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\psi(0) = 0$.
A set $A \subset X$ is said to be ψ-hypomotonic with constant R if for some $\varepsilon > 0$ for any $x_1, x_2 \in A$ such that $\|x_1 - x_2\| \le \varepsilon$ and for any $p_1 \in N(x_1, A), p_2 \in N(x_2, A)$, such that $\|p_1\| = \|p_2\| = 1$ the following inequality is true

$$\langle p_2 - p_1, x_2 - x_1\rangle \ge -R\psi\left(\frac{\|x_2 - x_1\|}{R}\right),$$

where $N(a_0, A) = \{p \in X^* : \forall \varepsilon > 0\ \exists\ \delta > 0 : \quad \forall a \in A \cap \mathfrak{B}_\delta(a_0) \quad \langle p, a - a_0\rangle \le \varepsilon \|a - a_0\|\}$.
We denote by $\Omega_P(R)$ ($\Omega_N^\psi(R)$) the set of all closed proximally smooth (ψ-hypomotonic) sets with constant R in X.

Theorem 1 *Let X be a uniformly convex and uniformly smooth Banach space and $\psi \in \mathfrak{M}$, then the following conditions are equivalent:*
1) there exists $k > 0$ such that $\Omega_P(1) \subset \Omega_N^{k\psi}(1)$;
2) $\rho_X(\tau) = O(\psi(\tau))$ $(\tau \to 0)$.

Theorem 2 *Let X be a uniformly convex and uniformly smooth Banach space and $\psi \in \mathfrak{M}$, then the following conditions are equivalent:*
1) there exists $k_1 > 0, k_2 > 0$ such that $\Omega_N^{k_1\psi}(1) \subset \Omega_P(1) \subset \Omega_N^{k_2\psi}(1)$;
2) X is isomorphic to the Hilbert space.

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