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Some aspects of Variational Analysis and Applications

# Properties and applications of weakly convex functions and sets[1]

**Grigorii E. Ivanov**

Moscow Institute of Physics and Technology

g.e.ivanov@mail.ru

**Abstract:** Let $E$ be a real Banach space. We say that the subset $M$ of $E$ is a *quasiball* if $M$ is closed convex and 0 is in the interior of $M$. The *$M$-distance* from the point $x_0$ to the set $A$ is $\varrho_M(x_0, A) = \inf_{a\in A} \mu_M(x_0 - a)$, where $\mu_M(x) = \inf\{t > 0 \,|\, x \in tM\}$ is the Minkowski functional. For a set $A \subset E$ and a point $x_0 \in E$ the *$M$-projection* of $x_0$ onto $A$ is $P_M(x_0, A) = \big\{a \in A \big|\, \mu_M(x_0 - a) \le \varrho_M(x_0, A)\big\}$. The set of *unit $M$-normals* for a set $A \subset E$ at a point $a \in A$ is defined as $N^1_M(a, A) = \{z \in \partial M \,|\, \exists t > 0 :\ a \in P_M(a + tz, A)\}$. A set $A \subset E$ is called *weakly convex* with respect to (w.r.t.) the quasiball $M \subset E$ if $a \in P_M(a + z, A)$ for all $a \in A$, $z \in N^1_M(a, A)$. If $M = \{x \in E \,|\, \|x\| \le r\}$, then the $M$-projection is the metric projection, the class of weakly convex sets is exactly the class of $r$-proximally smooth sets, investigated by Clarke, Stern, Wolenski, Bernard, Thibault, Zlateva and others. Colombo, Mordukhovich, Goncharov and Pereira studied the properties of weakly convex sets w.r.t. a nonsymmetric bounded quasiball. For a function $\gamma : E \to \mathbb{R}\cup\{+\infty\}$ and a number $r > 0$ we consider the function $\gamma_r(x) = r\cdot\gamma\left(\frac{x}{r}\right)$. The *$\gamma$-predifferential* of a function $f : E \to \mathbb{R}\cup\{+\infty\}$ at a point $x \in \text{dom } f := \{x \in E \,|\, f(x) \in \mathbb{R}\}$ is defined by $\pi_\gamma f(x) = \{u \in \text{dom } \gamma \,|\, \exists r > 0 :\ (f \boxplus \gamma_r)(x + ru) = f(x) + \gamma_r(ru)\}$, where $f \boxplus g$ is the infimal convolution of functions $f$ and $g$. A function $f : E \to \mathbb{R}\cup\{+\infty\}$ is said to be *weakly convex* w.r.t. $\gamma : E \to \mathbb{R}\cup\{+\infty\}$ if $(f \boxplus \gamma)(x + u) = f(x) + \gamma(u)$ for all $x \in \text{dom } f$, $u \in \pi_\gamma f(x)$. Note that in a Hilbert space the weak convexity w.r.t. the function $\gamma(x) = \sigma\|x\|^2$ $(\sigma > 0)$ is equivalent to weak convexity studied by Vial and lower-$C^2$ property due to Rockafellar.

**Theorem 1.** *Let $\gamma : E \to \mathbb{R}\cup\{+\infty\}$ be a convex lower semicontinuous (l.s.) function, continuous at 0 and $\gamma(0) < 0$. Then the function $f : E \to \mathbb{R}\cup\{+\infty\}$ is weakly convex w.r.t. the function $\gamma$ if and only if its epigraph* epi $f$ *is weakly convex w.r.t. the quasiball* epi $\gamma$.

**Theorem 2.** *Let $\gamma : E \to \mathbb{R}$ be a coercive function, bounded on any bounded set, and uniformly convex on any convex bounded set. Suppose that a function $f : E \to \mathbb{R}\cup\{+\infty\}$ is l.s. and weakly convex w.r.t. $\gamma$, $\text{dom}\,(f \boxplus \gamma) \ne \emptyset$. The function $f$ is weakly convex w.r.t. $\gamma$ if and only if for any $r \in (0, 1)$ and $x_0 \in E$ the problem $\min_{u\in E}\Big(f(u) + \gamma_r(x_0 - u)\Big)$ is well posed (i.e. every minimizing sequence of this problem converges to the minimizer).*

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