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Recent trends in shape and topology optimization

Shape Differentials and Topological Semidifferentials: a Semidifferential Geometric Approach

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Abstract: In the past decades, several direct constructions of complete *metric spaces of shapes and geometries* (cf., for instance, M. C. Delfour and J.-P. Zolésio [3]) and, more recently, additional new ones (cf., M. C. Delfour [2]) have been given without appealing to the classical notions of atlases or smooth manifolds encountered in classical Differential Geometry. Since, at best, such spaces are groups, the issue of making sense of tangent spaces and differentials naturally arises not only for "differentiable" functions but also for some classes of "non-differentiable" functions.

In that context, the notion of differentiable function introduced by J. Hadamard [7] is especially interesting since it implicitly involves the construction of trajectories and tangent vectors to trajectories living in the space under investigation. His definition was relaxed by M. Fréchet [5] in 1937 by dropping the requirement that the differential be linear with respect to the direction or tangent vector. A vast litterature on differentials on topological spaces followed (cf., for instance, the surveys of V. I. Averbuh and O. G. Smoljanov [1], M. Z. Nashed [9]). The definition of Fréchet was further relaxed to the notion of semidifferential which nicely handles convex and semiconvex functions while preserving two essential properties of the classical diferential calculus: the continuity of the function and the chain rule for the composition of functions.

The relaxation of the original definition has far reaching consequences. For a function $f : A \to B$ between two sets A and B, the semidifferential is no longer required to be linear. De facto, this relaxes the requirement that the tangent spaces in each points of A and B be linear spaces. As a result, it is sufficient to consider tangent cones to A and B such as the Bouligand's tangent cone to make sense of semidifferentials. Shortcircuiting the requirement of a smooth manifold makes it possible to directly study the tangent cones to metric spaces of shapes and geometries.

In this paper we review classical and new results. In particular, we show that some currently available metric spaces only have tangent cones made up of elements related to the "currents" introduced by H. Federer and W. H. Fleming [4]. In that perspective, it is remarkable

that the ground-breaking notion of *topological derivative* of J. Sokołowski and A. Zòchowski [10] is in fact a semidifferential on the metric space of characteristic functions and that the tangent space (of admissible directions) contains not only elements that create holes but also "currents" leading to topological perturbations along curves and surfaces that can break the connectivity of the set.

References

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