Bayesian optimal control for a non-autonomous stochastic discrete time system

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Abstract. The main objective of this article is to develop Bayesian optimal control for a class of non-autonomous linear stochastic discrete time systems with a random horizon of a control. By taking into consideration that the disturbances in the system are given by a random vector with components belonging to an exponential family with a natural parameter, we determine the Bayes control as the solution of a singular linear system. In addition we extend these results to generalized linear stochastic systems of difference equations.

Keywords: Bayes control, optimal, singular system, disturbances

Linear stochastic discrete time systems (or linear matrix stochastic difference equations), are systems in which the variables take their value at instantaneous time points and have a random horizon of a control. With the development of the digital computer, the stochastic discrete time system theory plays an important role in control theory. Stochastic discrete time systems have many applications in economics, physics, circuit theory, and other areas, see [1], [2], [3], [5]. Bayesian optimal control is a very important chapter for these kind of systems and thus many authors have studied it, see [5], [?], [7], [8], [4]. We consider the following non-autonomous linear stochastic discrete time system

$$\overline{x}_{n+1} = \alpha_n \overline{x}_n + b_n \overline{u}_n + c_n \overline{v}_n, \quad \forall n = 0, 1, ..., N-1. \tag{1}$$

Where $\overline{x}_n \in \mathbb{R}^m$ is the state of the system, $\overline{u}_n \in \mathbb{R}^m$ is the control, $\overline{v}_n \in V \subseteq \mathbb{R}^m$ is the disturbance at time n and α_n, b_n, $c_n \in \mathbb{R}^{m\times m}$. The horizon of the control N is a random variable, independent of $\overline{v}_0$, $\overline{v}_1$,..., with the known distribution given by $P\{N = k\} = p_k, \quad \forall k = 0, 1, ..., M, \quad \sum_{i=0}^{M} p_k = 1, \quad p_M \neq 0.$

Without loss of generality we can assume that $\overline{v}_n$ has the form

$$\overline{v}_n = (v_n^1, v_n^2, \ldots, v_n^k, 0, \ldots, 0)^T.$$

The vectors $\overline{v}_n$, $n = 0, 1, ..., M$ are independent, identically distributed random vectors with the components v_n^i, $i = 1, 2, ..., k$, having the distribution belonging to an exponential family (they may also belong to different families for different i), dependent on the unknown parameter λ_i, $i = 1, 2, ..., k$. Throughout the article, the following notations are used, $X_n = (\overline{x}_0, \overline{x}_1, ..., \overline{x}_n)$, $U_n = (\overline{u}_0, \overline{u}_1, ..., \overline{u}_n)$, $U^n = (\overline{u}_n, \overline{u}_{n+1}, ..., \overline{u}_M)$, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k, 0, ..., 0)^T$. For convenience U_M will be denoted by U and called a control policy. It is assumed that at time n both X_n and U_{n-1} are given and the control $\overline{u}_n$ is based on this information. Therefore before any data are obtained, the control $\overline{u}_n$ is a random vector determined by the random disturbances $\overline{v}_0$, $\overline{v}_1$, ..., $\overline{v}_{n-1}$. Let us assume that the cost of a control for the given control policy U (the loss function) is given by

$$L(U, X_N) = \sum_{i=0}^{N} (\overline{y}_i^T s_i \overline{y}_i + \overline{u}_i^T k_i \overline{u}_i),$$

where $\overline{y}_i = \begin{pmatrix} \overline{x}_i \\ \cdots \\ \overline{\lambda} \end{pmatrix} \in \mathbb{R}^{2m}$, $s_i \in \mathbb{R}^{2m\times 2m} \geq 0_{2m,2m}$, $\forall i = 0, 1, ..., M$. With $0_{i,j}$ we will denote the zero matrix $i \times j$. The risk connected with the control policy U, when the parameter $\overline{\lambda}$ is given, is defined as follows $R(\overline{\lambda}, U) = E_p E_{\overline{\lambda}} L(U, X_N)$. For the prior distribution π of the parameter $\overline{\lambda}$ the expected risk r, associated with π and the control policy U, is $r(\pi, U) = E_\pi[R(\overline{\lambda}, U)]$, where $E_p, E_{\overline{\lambda}}$ are the expectations with respect to the distributions of N and random vectors $\overline{v}_0, \overline{v}_1, ...$ (when $\overline{\lambda}$ is the parameter), E_π and E are the expectations with respect to the distribution π and to the joint distribution $\overline{v}_n$ and $\overline{\lambda}$, respectively. Let the initial state $\overline{x}_0$ and the distribution π of the parameter $\overline{\lambda}$ be given. A control policy U^* is called the Bayes policy when

$$r(\pi, U^*) = \inf_{U \in \wp_\pi} r(\pi, U),$$

where $\wp_\pi$ is the class of the control policies U for which the exists $r(\pi, U)$. Let the disturbances u_n^i, $n = 0, 1, ...$ belong to an exponential family with a natural parameter λ_i, $i = 1, 2, ..., k$, respectively. To this family belong distributions as the binomial, normal, gamma and Poisson, which are very frequently considered in practice.

The Bayes control for the conjugate prior distribution π of the parameter $\overline{\lambda}$ as the solution of *a singular linear system* is derived. An example of the Bayes control of a class of generalized linear stochastic discrete time systems is studied.

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